COMP 3804 – Design and Analysis of Algorithms
Assignment 4

Due: April 7, 2017 at 23:55

- Your solutions should be submitted online on cuLearn in the form of a single PDF file.
- Your answers should be precise, concise and clear. All algorithms should be given in pseudocode.
- Every part of every theory question is worth 2 marks. The grading scheme is 2 points for a correct answer, 0 for a completely incorrect answer, and 1 point for something in-between. The implementation part (question 1e) is worth 5% of the total grade.
- If you use any source other than the lecture notes, you **must** reformulate the information in your own words **and** cite the source. Failure to follow these rules will be treated as plagiarism.

1. Prim's algorithm computes a minimum spanning tree of a connected, undirected graph $G = (V, E)$ in which each edge (u, v) has a positive weight $wt(u, v)$. It works as follows:

    Algorithm PRIM'S ALGORITHM(G)
    $T \leftarrow \{v\}$ (arbitrary vertex)
    while $|T| < |V|$ **do**
    add the shortest edge (a, b) with $a \in T$ and $b \notin T$ to T
    return T

    (a) Prove that the output of Prim's algorithm is connected.
    (b) Prove that the output of Prim's algorithm is a minimum spanning tree.
    (c) Describe in detail how you would implement the body of the while-loop. How do you find the shortest edge? What data structure(s) do you use? How are these initialized and updated? Give pseudo-code where relevant.
    (d) Analyze the running time of Prim's algorithm, using the implementation details described in 1c. For full marks, your running time should be $O(|E| \log |V|)$ or better.
    (e) Implement an algorithm to compute the minimum spanning tree of a graph in the accompanying zip file.

2. In this question, you will show that two problems have essentially the same difficulty. In formulating the 3-SAT instance, you may use $\geq_k (x_1, x_2, \ldots, x_m)$ as notation for a 3-CNF constraint with $O(nk)$ clauses that specifies that at least k of the variables x_1 through x_m should be true.

    (a) Prove that 3-SAT $\leq_P$ INDEPENDENT SET.
    (b) Prove that INDEPENDENT SET $\leq_P$ 3-SAT.

3. The EULER CYCLE problem is defined as follows: given an undirected graph G, does it have a cycle that uses every edge exactly once?

    (a) Prove that EULER CYCLE $\leq_P$ HAMILTONIAN CYCLE.
    (b) One of your classmates claims to have found a polynomial-time algorithm that solves EULER CYCLE. If this is true, what implications would this have for HAMILTONIAN CYCLE?

4. Let $G = (V, E)$ be a directed graph where each edge $e = (u, v)$ has weight $wt(u, v) > 0$ or $wt(u, v) < 0$ (i.e. no edge has a weight of zero). The ZERO CYCLE problem asks whether G has a directed cycle such that the sum of the weights of the edges in the cycle is zero.

   (a) Prove that this problem is in NP.

   (b) Prove that this problem is NP-hard.